1971

A nonlinear mathematical programming approach to activity analysis and decentralized planning procedures.

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Monterey, California ; Naval Postgraduate School

A NONLINEAR MATHEMATICAL PROGRAMMING APPROACH TO ACTIVITY ANALYSIS AND DECENTRALIZED PLANNING PROCEDURES

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THESIS

A NONLINEAR MATHEMATICAL PROGRAMMING APPROACH TO ACTIVITY ANALYSIS AND DECENTRALIZED PLANNING PROCEDURES

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September 1971

A Nonlinear Mathematical Programming Approach to Activity Analysis and Decentralized Planning Procedures

by

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Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
September 1971

ABSTRACT

A review of some of the current literature pertaining to this thesis is conducted. The Created Response Surface Technique of solving a class of nonlinear mathematical programs is presented. The theoretical interpretations of the primal and dual formulations of the technique in an activity analysis context are developed. The applicability of these interpretations to the neoclassical theory of the firm and the contemporary organization theory is indicated. Computational experience in solving well defined numerical problems is also indicated. Several linear models of organizational decentralized planning are reviewed. A nonlinear model of decentralized planning procedures is proposed and solved using the Created Response Surface Technique.

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ACKNOWLEDGEMENT

The author wishes to express his gratitude to Professor Carl R. Jones of the Operations Research and Administrative Sciences Department of the Naval Postgraduate School for his assistance and guidance throughout the preparation of this thesis. Professor Jones' willingness to listen and helpful suggestions were conducive to expanding rudimentary analytic tools in broadening the author's concepts of organizational behavior.

The author wishes to thank Professor James K. Hartman of the Operations Research and Administrative Sciences Department of the Naval Postgraduate School for his assistance in the mathematical programming aspects of this thesis.

I. INTRODUCTION

A. REVIEW OF NEOCLASSICAL THEORY OF THE FIRM

In this thesis the neoclassical theory of the firm and contemporary organization theory are extended by the application of a nonlinear programming technique known as the Created Response Surface Technique (CRST). For ease of understanding, the reader is expected to have a basic grasp of the neoclassical theory of the firm, the static economic theory of market equilibrium, and from the mathematical disciplines the classical theory of optimization including Lagrange multipliers and mathematical programming. This introduction contains a brief review of the neoclassical theory of the firm and a description of the background motivating this thesis.

In neoclassical economic theory a firm, or producer, is an economic agent who consumes some commodities and produces other commodities. It is usual to embed the producer in an economy characterized by scarce resources. By the interaction of consumers and producers in markets, commodities which are goods or services are valued as scarce, free, or noxious. More specifically, the firm technologically transforms inputs, factors of production which are purchased in markets, into outputs, products which are sold in markets. In making choices the firm considers consumers' and other producers' demand for its outputs, the state of production technology and the behavior

of the market where it purchases its inputs and sells its outputs. The firm is usually assumed to behave as if it maximized its profit. In this case profit is the difference between the total revenue received from the sale of its outputs and the total costs incurred in the production of those outputs.

In a perfectly competitive economy a market mechanism establishes prices for all commodities including the firms inputs and outputs. The firm is assumed to accept these prices as given datum determined in the market (atomistic competition) and to pursue some form of optimizing behavior. This behavior can be described by three different behavioral models. First, there is the maximization of physical output subject to a cost constraint. Second, there is the economic dual formulation of that model, the minimization of total cost subject to an output constraint. And finally, there is the maximization of profit, which under suitable conditions embeds the previous two formulations as subproblems.[1]

If the production, cost and profit functions are sufficiently well behaved,[2] mathematical programming techniques can be applied to these models. The application of mathematical programming techniques to these problems usually

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[1] Henderson, J. M., and Quandt, R. P., Microeconomic Theory, p. 53, McGraw-Hill, 1958.

[2] Kuhn-Tucker Constraint Qualification, Reference, Appendix A, paragraph 9.

results in decision rules which characterize an optimum. If the functions or their approximations are sufficiently mathematically tractable, numerical solutions can be achieved.

B. DECENTRALIZATION PLANNING PROCEDURES

1. General Model

In neoclassical theory the firm is modeled as if it were an homogeneous entity. Mathematical functions represent the firm's aggregate behavior *ex cathedra* in the economy. The decision rules derived from these models indicate the necessary and sufficient conditions to achieve optimal behavior.

Several models have been proposed which describe how the firm pursues the optimum behavior once the necessary and sufficient conditions for that optimum behavior have been established. One model proposed by E. Malinvaud [1]* considers the firm's pursuit of optimum behavior to be a sequential decentralized planning procedure. This model is representative of a general class of decentralization models and will be presented to introduce the basic concepts and the pertinent assumptions underlying such decentralized planning models.

This model characterizes a firm which is sufficiently large to have such organizational subunits as a central agency and at least one production agency, denoted

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* References in brackets may be found in the List of References.

by j where j = 1,...,J. It is supposed that the central agency is responsible for formulating a plan for optimizing the behavior of the entire firm. Assume that the central agency can forecast the aggregate demand for the firm's products over a time period in which technology may be considered constant. Further suppose that the central agency knows the quantities of available inputs and the entrepreneur's preferences, which may be represented by a utility function, but it does not know the specific technical information which describes the actual production process. This specific information is known by production agencies, but each specific production agency does not know the entrepreneur's preferences, the availability of inputs, or the production levels of other production agencies, if they exist in the firm. Clearly, some method should be used to allow the production agency to provide its specific technological information in the central agency's formulation of the plan. Although the formulation of the plan is in reality a continuous process, it is assumed that this plan will be created over a time period in which these exchanges of information between the central agency and the production agencies will form an iterative procedure.[3]

This iterative procedure is assumed to be accomplished in discrete stages. Each stage is denoted by s,

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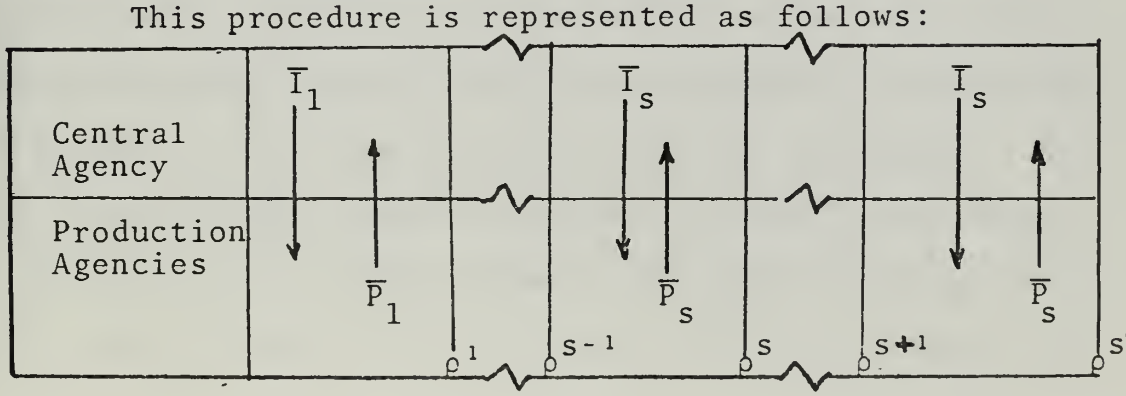
[3] This conclusion presupposes some form of administrative regulation governing how information is exchanged and some requirement for minimum content of the information exchanged.

$s = 1,...,S$. The central agency provides "prospective indices", $\overline{I}_s$, a vector consisting of J components at stage s. Each component of this vector is the "prospective index" to each of the J production agencies at the s^{th} stage. A prospective index is essentially a request for technological information. It's form may be a production quota, resource availability, or a budget constraint. For example, the central agency may be asking, "What can you, the j^{th} production agency, produce if you have "X" dollars of budget available?" Each prospective index requires some response from each of the production agencies. In return, the production agencies provide a "proposal", $\overline{P}_s$, a vector consisting of J components at stage s. Each component of this vector is a "proposal" from one of the J production agencies. A proposal implicitly provides the specific technical information embedded in the j^{th} production agency. For example, the j^{th} production agency may be responding, "Given "X" dollars of budget, I can produce "Y" units of my specific output." Each mutual exchange of information is a stage in the formulation of the "plan". The vector, ρ^s, will denote the "plan" after the s^{th} stage.

This procedure is represented as follows:



where,

$$\overline{I}_s = \begin{bmatrix} I_1 \\ \cdot \\ \cdot \\ \cdot \\ I_j \\ \cdot \\ \cdot \\ \cdot \\ I_J \end{bmatrix} \qquad \overline{P}_s = \begin{bmatrix} P_1 \\ \cdot \\ \cdot \\ \cdot \\ P_j \\ \cdot \\ \cdot \\ \cdot \\ P_J \end{bmatrix}$$

2. Characteristics

Analogous to the characteristics of the simplex algorithm of linear programming, any iterative procedure can be studied to understand which characteristics it exhibits. The following is a partial list of possible characteristics of an iterative procedure. First, the procedure could be well defined. That is, if at each stage of the process solutions exist to the "prospective indices", "proposals" and the "plan", the procedure is well defined. Secondly, the procedure could be monotonic. If the utility of the s^{th} iteration is at least as great as the utility of the $s\text{-}1^{st}$ iteration, the process will be monotonic. Next, the procedure could be convergent. If, as S increases beyond bound, the utility of the s^{th} iteration approaches some least upper bound of the set of utilities defined over all feasible plans, the procedure will be convergent. Finally, the procedure could be finite. If the "plan" after the s^{th} iteration is the optimal plan for some finite number S, the procedure will be finite. If the procedure is

not finite, it is assumed that the central agency will establish an upper bound on the number of iterations of the process. In reality, the time and cost of each exchange of information will tend to constrain the process. Some optimization technique is indicated and would be appropriate. This optimization technique will not be discussed, but it may provide an area for future interest.

3. Walrasian Tatonnement Process

As an example of iterative planning procedure, consider the neoclassical Tatonnement (recontracting) process of static market equilibrium. A perfectly competitive market might consist of an auctioneer, customers, and producers (production agencies) who are assumed to produce only one unique output. The rational behavior of the consumers and producers yields market equilibrium (the plan) by the Walrasian price adjustment mechanism. The price adjustment mechanism acts as a stopping rule signifying the termination of the iterative procedure and achievement of the necessary and sufficient conditions for market equilibrium. The procedure starts when the auctioneer, real or pedagogical, provides a price vector (prospective index) of all the commodities in the market to consumers and producers. This initial vector may be based on past performance of the market or "judgment".

The consumers seek to maximize their individual utility subject to a budget constraint generated by the prices of commodities in their prospective commodity

bundles and their initial endowment. The quantities of goods and services the consumers are willing to buy at the existing prices form the aggregate demand at this iteration.

Concurrently, producers seek to maximize their individual profits. Since profit is a function of inputs and outputs, the existing prices allow firms to determine the quantities of their respective outputs they are willing to produce. The vector of these quantities (proposal) is the aggregate supply at this iteration.

If consumer demand for the j^{th} product at the s^{th} iteration, denoted by D_{sj}, exceeds production supply of the j^{th} product of the s^{th} iteration, denoted by S_{sj}, or vice versa for any of the commodities, the auctioneer will adjust the appropriate prices to account for the excess demand or supply. The iterative procedure continues until the market is cleared, $\overline{D}_s - \overline{S}_s = \overline{0}$, at an equilibrium price, indicating that the plan is optimal.

II. REVIEW OF LITERATURE

Activity analysis is the synthesis of mathematical programming and the economic theory of the firm. When mathematical programming techniques are applied to well defined behavioral models of the firm, the optimum solution to the programming formulation of that model yields appropriate managerial decision rules which characterize that optimum solution. Likewise, if numerical data is used in a practical model of the firm, the optimum solution to the programming formulation of that model will indicate how that optimum solution may be achieved with that data.

Historically the theoretical and practical applications of activity analysis to well defined resource allocation problems appear to have developed as a hybrid of research in both disciplines. The introduction of a mathematical programming technique is generally followed by the application and interpretation of that technique in an economic context.

Whether or not a cause-effect relationship actually exists is not conjectured. Rather, this apparent sequence will be used to provide topic continuity in reviewing some of the principal research efforts in areas which directly relate to this thesis.

The first known model of an economy which introduced technological production in characterizing a general economic

equilibrium was developed by J. von Neumann [2] in 1937. The model treated technological coefficients, outputs, and production processes as known entities. Activity levels, prices, the coefficient of expansion of the entire economy (scale of the economy) and the interest factor of the economy were considered unknown entities. The model could accomodate the concepts of joint production, capital goods depreciation, and scarce or free goods. Employing the Minimax criterion of Games Theory, von Neumann proved the existence of at least one solution to the coefficient of expansion and the interest factor as functions of prices and activities. At optimality, the solution to the model uniquely equates the coefficient of expansion to the interest factor. The equality of these implicit functions of prices and activity levels characterize the equilibrium conditions of the economy in perfect competition.

During World War II the allocation of resources in multi-theatre military operations and in the domestic economy generated interest in mathematical models of definitive planning procedures. A landmark in the research effort in linear function analysis was G. B. Dantzig's [3] introduction of the simplex algorithm in 1947.

T. C. Koopman's [4] immediate appreciation for the variety of allocation problems which could be modeled by procedures similar to the simplex algorithm resulted in the emphasis on activity analysis at the Cowles Commission for Research in Economics Conference of 1951. The primary

purpose of this conference was to consolidate the results of various independent research efforts which had been addressing similar problems in an activity analysis context.

In a recent book, D. C. Vandermeulen [5] describes the foundations and origin of the simplex algorithm and its application to the neoclassical theory of the firm. The iterative procedure of the algorithm is interpreted as a model of the rational economic behavior of an hypothetical entrepreneur. The decision rules governing optimization of a defined production objective are demonstrated. The economic analysis of complementary slackness conditions and the applicability of the dual formulation of the model are presented.

In 1950 H. Kuhn and A. Tucker [6] formulated the necessary and sufficient conditions which characterize the saddle point of a two person zero sum game and applied those conditions to a class of nonlinear inequality constrained programming problems. Under appropriate differentiability and nonnegativity assumptions, it was shown that the maximum of a concave objective function over a convex set defined by inequality constraints was equivalent to the saddle value of the Lagrangian function of that programming problem. The interpretation of the Kuhn-Tucker results when applied to economic theory was consistent with concurrent economic research efforts.

In 1961, R. Pfouts [7] applied the Kuhn-Tucker necessary conditions to a convex problem formulation of cost and

production in the multi-product firm in a perfectly competitive economy. In this formulation, the multi-product firm was modeled as a homogeneous economic entity rather than an aggregation of single product firms, as in the Hicksian model. This approach characterized each product as competing for a portion of the total amount of fixed factors of production (long run sense) assessing a "set-up" or transfer cost for changes in product mix which required corresponding changes in the allocation of the fixed factors between periods (short run). Levels of the variable factors of production (short run sense) were assumed to change with the output level.

The model seeks to minimize a total cost function consisting of variable, fixed, and transfer cost terms constrained by technology and continuity of fixed factor allocation to specific products. The convex problem is solved by applying the Kuhn-Tucker necessary conditions yielding the following results at equilibrium. Consistent with the Hicksian model, the imputed value of marginal output must be at least as great as the marginal costs of that output. But additionally, the imputed value of the reallocation of fixed resources from one product to another must at least be equal to the value of the increased marginal product resulting from the change minus the cost associated with the marginal transfer of those fixed resources. Thus, only when an excess of all fixed factors exists can the multi-product firm be considered an aggregation of single product firms.

Four years later, T. Naylor [7] expanded Pfouts' analysis of the multi-product firm in a competitive economy to a monopolistic-monopsonistic economy. Maintaining Pfouts' concept of a multi-product model, Naylor optimizes the economic dual of that model. At equilibrium the decision rules characterizing the optimum substantiated Pfouts' previous conclusions.

In 1958, G. Dantzig and P. Wolfe [9] introduced a decomposition principle for linear programs. The principle is applicable to programs which have some constraints which "bind" together various sets of constraints which are independent of each other. The original problem is decomposable into subprograms, a characterization of each independent constraint, and a master program, an aggregation of the subprograms. A full master program is formed from the convex combinations of all the extreme points of the feasible regions of each of the independent constraints. Each basic feasible solution to the master problem generates dual variables. A function of the dual variables establishes an optimality criterion for the value of the master program objective function. If this criterion is not met for any one or more of the independent subproblems, the subprogram for that constraint set is optimized providing a new vector to enter the basis and generating a new basic feasible solution to the master program. This procedure continues iteratively until the optimality criterion is achieved for all independent subprograms or it is evident an optimal solution does not exist.

In 1958, L. Hurwicz [10] describes the "greed process" as a model of information decentralization for a class of decomposable economic environments. In this context decomposable means free of external (dis)economies of scale. The "greed process" embodies the intuitive concept of decomposition because each economic agent is assumed to be concerned with his actions alone, even though this may be against his economic best interests in the operation of the system. Each agent lacks any knowledge of the internal functioning of other agents in the economy. Hurwicz proves that the "greed process", the interaction of these independent economic agents, achieves Pareto optimal conditions but requires more information to achieve these conditions than the neoclassical competitive market mechanism. Although this "greed process" is shown to be applicable to a broader class of economic environments, it lacks the informational efficiency of the perfectly competitive mechanism which decentralizes the economy by prices alone.

In 1963, Y. Ijiri [11] applied a special form of linear programming, "goal programming", to accounting models of the firm. Goals may take the form of lower bounds on production or upper bounds on input usage. The procedure emphasizes the decomposition of definitive management goals, which correspond directly to profit, into a set of subgoals which are more tractable for subordinate agencies within the firm. Management seeks to minimize the firms deviation from the firm's goals. The deviations from the firm's goals

are an aggregate of the subordinate agencies' deviation from their respective independent goals. If subordinate goals are compatible, a case in which all goals can be satisfied simultaneously, specific goal levels and deviations from those levels are sufficient information to decompose the firm and iteratively achieve an optimum solution. If subordinate goals are incompatible, a case in which all goals cannot be satisfied simultaneously but some goals must be satisfied sequentially, Ijiri proposes a "preemptive priority", an ordering of goal achievement. Further, if there are several goals of the same order, a weighting scheme can be used to provide relative differentiation among goals of equal priority. Utilizing a procedure similar to the simplex algorithm of linear programming, the goals with the "lowest cost", i.e., highest priority and weight, sequentially enter the basis until an optimum solution for the existing priority and weighting scheme is achieved.

Recently, T. Ruefli [12] analyzed resource allocation in a three level firm by utilizing a generalized goal decomposition model. The organization consists of a central agency, management and operational units. Each level is vertically interrelated by a specific information flow. Levels of organization are assumed to be horizontally independent. The central agency seeks to maximize the imputed value of scarce resources subject to the allocation of those resources to different managements. Managements

seek to minimize the deviations from these specified goals subject to the productional relationship of their independent operational units. Managements pass the shadow prices of their goals to subordinate operating units and the central agency. Each operating unit minimizes it's management's imputed value of their (each operating unit's) production activity subject to each operating unit's particular technology. Operating units propose activity levels at that shadow price to their respective managements. Concurrently, the central agency determines revised goals. Managements sequentially determine revised shadow prices given these new goals and new production activities. The procedure continues iteratively until a desired optimum is achieved.

In 1959, C. Carrol [13] proposed the Created Response Surface Technique, a penalty function approach to the solution of nonlinear constrained mathematical programs which modeled complex industrial processes. The technique "created" a form of weighted penalty function from the original objective function and original constraints. The procedure seeks to optimize the unconstrained created function in a sequence of steepest ascent steps. Each iteration of the procedure seeks the optimum to an artifical objective surface. This optimum in turn generates another artificial objective surface whose optimum value is sought. As the number of iterations increase beyond all bound, the artifical objective surface approaches the surface of the original objective function and the artifical optimum approaches the original optimum. Throughout the procedure

feasibility is assured by a weighted "boundary-repulsion" term which is sequentially reduced at each iteration.

In 1961, P. Wolfe [14] formulated a dual program for a nonlinear, convex, differentiable, primal, objective function minimized over a convex constraint region. Applying the Kuhn-Tucker Equivalence Theorem conditions to the Lagrangian Function of the primal problem, Wolfe proved the existence of the dual solution at optimality. Also at optimality the values of the primal and dual objective functions are equal.

In 1968, M. Balnski and W. Baumol [15] interpreted Wolfe's dual formulation to a nonlinear primal problem of profit maximization constrained by technology in a competitive economy. Maximization of a concave profit function subject to concave technological constraints resulted in the dual minimization of a Lagrangian function consisting of a fixed profit function, the value of unused resources and the total marginal opportunity losses of all outputs. The minimization of the dual objective function minimized the value of unused resources and the total marginal opportunity losses of all outputs. Regrouping of terms in the dual objective function provides a mathematical interpretation of economic rent which is the difference between total profit and the imputed value of all inputs. The existence of economic rent is attributed to the concave object function representing diminishing returns. This assumption indicates that the imputed value of inputs should be less than total profit.

In 1963, A. Fiacco and G. McCormick [16, 17, 18] developed a modification of Carroll's Created Response Surface Technique which solves a constrained, nonlinear mathematical program in a sequence of unconstrained minimization problems. The original programming problem is to minimize a convex objective function subject to concave constraints. A penalty function is formed from the original objective function and constraints. The sequential unconstrained minimization of this form of penalty function approaches the optimal value of the original problem by iterative gradient search methods. A "boundary repulsion" term insures feasibility from an interior starting value. Primal-dual feasibility, recommendations based upon computational experience, and the theoretical structure of the technique are presented.

III. THE CREATED RESPONSE SURFACE TECHNIQUE

A. PROCEDURE DESCRIPTION

In activity analysis, mathematical programs are utilized to model the economic activity of the firm. Customarily, the model consists of an objective function, a function of endogenous variables under the control of management, and constraints, levels of exogenous variables which are determined from the economic environment. Assuming neoclassical rational behavior, the objective function is maximized or minimized, as appropriate within the bounds defined by the constraints.

If the objective function and constraint equations are linear functions, linear programming techniques may be used to determine the appropriate decision rules or levels of endogenous variables. If the objective function and constraint equations are nonlinear but separable functions, separable programming techniques may be used to solve the program. Similarly, quadratic programming methods may be used if the objective function is quadratic and the constraint equations are linear.

There are few developed techniques of solution for those models which are not classified as linear, quadratic, or separable. However, an auxiliary function method has been developed for this class of models. The general objective of this approach is to transform the original constrained problem into an unconstrained "auxiliary function" which may be optimized by several available techniques. The

auxiliary function incorporates the objective function and constraint equations of the original problem.

One particular class of auxiliary functions consists of the original objective function and a penalty term which algebraically contributes a penalty for violation of the constraints of the original problem. This penalty term may be logarithmitic or reciprocal-linear depending on the particular model. The intent of this penalty function approach is to insure that the number of infeasible solutions is decreased as the procedure approaches the optimum of the auxiliary function. The optimum of the auxiliary function will be the optimum of the original objective function if the procedure is convergent.

The Created Response Surface Technique [13] is one form of the penalty function approach to the auxiliary solution of a constrained mathematical program.

As an explanation of this technique, consider the following problem:

$$\text{(A)}\qquad \begin{aligned} &\text{MAX:} && f(x_1 \cdots x_j \cdots x_J) \\ &\text{S.T.} && g_i(\bar{x}) \geq k_i \quad i = 1,\ldots,I \end{aligned}$$

where,

$f(\bar{x})$ is a concave function with continuous first and second partial derivatives.

$g_i(\bar{x})$ for $i = 1,\ldots,I$ are concave functions with continuous first and second partial derivatives.

The constraint region formed by the $g_i(\bar{x})$ for $i = 1,\ldots,I$ is a convex region. If the nonnegativity of the dependent variables is required, these would be included as additional constraint equations, ($i = I+1,\ldots,I+J$), in the problem.

The created response function problem for (A) is:

(B)

$$\text{MAX:} \quad P(\bar{x},r) = f(\bar{x}) - r \sum_{i=1}^{I} \frac{w_i}{g_i(\bar{x}) - k_i}$$

The elements of this function are:

1. Created response function: $P(\bar{x},r)$
2. Objective function: $f(\bar{x})$
3. i^{th} constraint equation: $g_i(\bar{x}) \geq k_i$
4. i^{th} constraint level: k_i
5. i^{th} reciprocal deviation: $1/g_i(\bar{x}) - k_i$
6. i^{th} subjective weighting factor: w_i ; $w_i > 0$
7. Penalty term: $\sum_{i=1}^{I} \frac{w_i}{g_i(\bar{x}) - k_i}$
8. Penalty term weighting factor: r ; $r > 0$.

The original constrained maximization problem has been transformed to an unconstrained maximization problem. Several methods are available to perform this unconstrained maximization. The second order gradient search method will be used because of its proven computational efficiency.[4]

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[4] Fiacco, A. and McCormick, G., "Computation Algorithm For the Sequential Unconstrained Minimization Technique for Nonlinear Programming," Management Science, Vol. 10, No. 4, p. 607, July 1964.

Intuitively, the maximization of the created response function should decrease the value of the weighted penalty term relative to the value of the objective function. The procedure seeks to maximize the function and avoid the bounds of the constraint region simultaneously. If any one constraint approached its binding value, $g_i(\overline{x}) = k_i$, the penalty term would increase beyond bound. As the weighting factor of the penalty term, r, is sequentially reduced to zero, the procedure will allow constraints to become "more binding" in the sense of approaching their respective boundary values. In the limit, the value of the optimum of the created response function approaches the value of the optimum to the original problem.

Assume that the I subjective weighting factors have been assigned. Starting the procedure with an initial feasible point, $\overline{x}_1$, and initial value of r, r_1, a response surface, $P(\overline{x},r_1)$, is created. The second order gradient search method seeks the optimum value of that response surface starting at $\overline{x}_1$. The optimum value of $P(\overline{x},r_1)$ yields a new starting point, $\overline{x}_2$. $P(\overline{x},r_2)$, where $r_2 < r_1$, is a new created response surface whose optimum value, $\overline{x}_3$, is found by using second order search starting at $\overline{x}_2$. Then r_3 is chosen with $r_3 < r_2$ and the process continues. Each new r value creates a new response surface until as,

$$\lim_{\ell \to \infty} r_\ell \to 0$$

the optimal value of the ℓ^{th} created response surface approaches the optimum value of the original problem.[5]

B. PRIMAL FORMULATION

1. Practical Application

The Created Response Surface Technique exhibits several characteristic properties which indicate the feasibility of practical applications of the technique in activity analysis. The following description of these properties will be used to sketch the procedure's potential application and to state these properties without the rigorous proofs developed in References [16, 17, 18].

The Created Response Surface Technique could be used to solve models of complex production processes without requiring that linear, quadratic, or separable conditions be met. Linear models assumed constant returns to scale. Nonlinear models could permit production functions which demonstrate increasing, constant, or decreasing returns to scale over the range of the endogenous variables. Linear models assumed production processes were independent and in fixed proportion; nonlinear models could introduce the interactive effects of different production processes at various levels of mix which could not be described by quadratic or separable functions.

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[5] Fiacco, A. and McCormick, G., "The Sequential Unconstrained Minimization Technique for Nonlinear Programming, A Primal-Dual Method," Management Science, Vol. 10, No. 2, p. 361, January 1964.

The convergence of the optimal values of (A) and (B) is assured in the limit. The stability, which is the local improvement of the created response function at each iteration, provides assured improvement of the objective function for a finite number of iterations of the procedure. A criterion for local improvement from an initial starting point may be achieved.

Constraint feasibility is assured in all iterations of the procedure. Although this characteristic may require an increased number of iterations to reach a solution which may be achieved in less iterations by permitting temporary infeasibility, the temporary infeasibility may be economically uninterpretable in the context of the original problem.

By utilizing gradient search techniques to optimize the created response function, local improvement for each value of r is achieved. Assuming that the response surface is well behaved, changes in the values of the endogenous variables should reflect local changes during each iteration. These endogenous variables may represent physical production processes. Minor changes in these processes represent "local" improvement which may be less costly to introduce them than massive changes required if the global optimum is achieved indirectly without intermediate steps. The intermediate "optimum" values of the variables generated by each iteration of the search technique may facilitate making physical changes in the appropriate production process.

The procedure establishes primal-dual bounds on the optimal solution of the original problem at each iteration. Dual feasible points are generated from the optimal value of each created response function. The value of the dual objective function becomes an upper bound, and the value of the created response function becomes a lower bound for the optimal value.[6] As the number of iterations increase beyond bound both upper and lower bounds converge to the optimal value of the original objective function. This property is useful for establishing a criterion for achieving a desired accuracy for an approximation to that optimal solution since the number of iterations usually are restricted by transaction cost or time considerations.

The procedure may be useful when applied to an original problem which does not observe the requisite convexity assumptions. Practical applications to nonconvex problems indicate excellent results.[7]

2. Theoretical Application

Rational economic behavior, in a neoclassical contest, is optimizing behavior. The entrepreneur is modeled as maximizing profit or output or minimizing cost. Embedded in a perfectly competitive economy, the neoclassical entrepreneur possesses perfect information as well as the assumed computational capacity to compare all the alternatives

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[6] Ibid, p. 363.

[7] Fiacco and McCormick, op. cit., p. 610.

generated by perfect information. He selects the one alternative that is optimal for his well defined utility function.

Rational economic behavior in a contemporary context, proposed by Simon and March, [19], is "satisficing" behavior. The contemporary manager does not possess perfect information, a well defined utility function, or the computational capacity of the neoclassical entrepreneur. The contemporary manager seeks to discover satisfactory alternatives which at least meet or perhaps exceed an established criterion. He seeks to choose an alternative from those discovered. Only on rare occasions will he seek an optimum alternative in the neoclassical sense because in the modern model information and computation are not costless.[8]

The Created Response Surface approach is adaptable to both the neoclassical and contemporary theories of the firm. Several standard economic interpretations of the terms of the created response function are presented to demonstrate this adaptability.

The objective function, $f(\overline{x})$, may represent any one of the many forms of functional descriptions of a firm's effectiveness which the firm desires to improve. $f(\overline{x})$ may describe profit, output, or some representation of efficiency or utility.

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[8] Simon, H. A. and March, J. G., <u>Organizations</u>, p. 140, 1958.

The constraint levels, k_i, may represent input resources or output goals. As an input constraint level, k_i, would be an upper bound on the resource usage. An an output constraint level, k_i, would be a lower bound or minimum acceptable production goal for the i^{th} product.

The function, $g_i(\overline{x})$, may represent the manner in which each of the I resources are utilized or goals are achieved in the production process. Grouped in matrix form those constraints which are upper bounded by input resource levels, $-g_i(x) \geq -k_i$ represent how some or all of the J production processes use the i^{th} input resource. Similarly, those constraints which are lower bounded by minimum production goals, $g_i(\overline{x}) \geq k_i$, represent how some or all of the processes interact to produce the i^{th} output goal.

The matrix is customarily structured in the following manner:

$$\begin{bmatrix} g_1(\overline{x}) \\ \vdots \\ g_k(\overline{x}) \\ -g_{k+1}(\overline{x}) \\ \vdots \\ -g_I(\overline{x}) \end{bmatrix} \geq \begin{bmatrix} k_1 \\ \vdots \\ k_k \\ k_{k+1} \\ \vdots \\ -k_I \end{bmatrix}$$

where,

$g_i(\overline{x})$ for $i = 1,\ldots,k$ is a functional representation of production.

$g_i(\bar{x})$ for $i = k+1,\ldots,I$ is a functional representation of resource utilization.

The subjective weighting factor for each of the I constraints embodies the decision maker's subjective concern for each constraint relative to the other constraints. In this sense, the decision maker's concern is an expression of the relative importance that the i^{th} constraint equation be binding. Interpreted as a positive priority number the lowest value w_i would depict the decision maker's highest subjective concern for that constraint compared to all other constraints. This explicit priority structure causes the Created Response Surface Technique to "accomodate" the higher priority constraint boundaries sooner in the iterative process.

This ordering may represent a form of utility function over a finite commodity space consisting of inputs and outputs. Assigned *a priori*, the revealed priorities describe how a decision maker trades off between the i^{th} input usage and the i^{th} constraint level, or between the i^{th} production and the i^{th} output goal.

The reciprocal deviation term for each of the I constraints is a measure of distance which corresponds to a reciprocal "amount" of deviation from each specified production goal or resource level. When the weighted individual reciprocal deviation terms are aggregated to form a penalty term, this concept of distance is not clear. In a two constraint input case, $i = 1,2$, the penalty term is

$$\sum_{i=1}^{2} \frac{w_i}{k_i - g_i(\bar{x})} = \frac{w_1(k_2 - g_2(x)) + w_2(k_1 - g_1(\bar{x}))}{(k_2 - g_2(\bar{x}))\,(k_1 - g_1(\bar{x}))} \; .$$

This measure of distances does not correspond to the metric measure of distance:

$$d_p = \left[\sum_{i=1}^{2} (k_i - g_i(\bar{x}))^P \right]^{1/P}$$

$p = 2$ is the usual Euclidean measure.

The penalty term weighting factor, r may represent the decision maker's concern for the penalty term relative to the objective function. In a production context, assume that $f(\bar{x})$ represents an internal index of productivity as a function of activity levels. The problem is to:

$$\text{MAX:} \quad f(\bar{x})$$

(C)

$$\text{S.T.} \begin{bmatrix} g_1(\bar{x}) \\ \vdots \\ g_k(\bar{x}) \\ -g_{k+1}(\bar{x}) \\ \vdots \\ -g_I(\bar{x}) \end{bmatrix} \geq \begin{bmatrix} k_1 \\ \\ k_k \\ -k_{k+1} \\ \\ -k_I \end{bmatrix}$$

$$\bar{x} \geq 0$$

where,

$f(\bar{x})$ is a concave function with continuous first and second partial derivatives.

$g_i(\bar{x})$ $i = 1,\ldots,k$ is a concave function with continuous first and second partial derivatives.

$g_i(\bar{x})$ $i = k+1,\ldots,I$ is a convex function with continuous first and second partial derivatives.

Applying the Created Response Surface Technique formulation the primal problem becomes:

$$\text{MAX:} \quad P(\bar{x},r) = f(\bar{x}) - r\left[\sum_{i=1}^{k} \frac{w_i}{(g_i(\bar{x}) - k_i)} + \sum_{i=k+1}^{I} \frac{w_i}{(k_i - g_i(\bar{x}))}\right]$$

where

$w_i > 0$ are assigned *a priori*

$r > 0$ such that $r_1 > r_2 > \cdots > r_e$.

Solving the Created Response Surface Technique formulation, the necessary conditions are:

$$\frac{\partial f(\bar{x})}{\partial x_j} = r_\ell \left[\sum_{i=1}^{k} \frac{-w_i}{(g_i(\bar{x}) - k_i)^2} \frac{\partial g_i(\bar{x})}{\partial x_j} + \sum_{i=k+1}^{I} \frac{w_i}{(k_i - g_i(x))^2} \cdot \frac{\partial g_i(\bar{x})}{\partial x_j}\right],$$

where

$\frac{\partial f(\bar{x})}{\partial x_j}$ represents the internal marginal index of productivity.

$\frac{\partial g_i(\bar{x})}{\partial x_j}$ $i = 1,\ldots,k$ represents the weighted marginal output technology.

$\frac{\partial g_i(\bar{x})}{\partial x_j}$ $i = k+1,\ldots,I$ represents the weighted marginal input technology.

The necessary conditions of the created response function generated for each value of r, r_ℓ, indicate that the marginal productivity of the j^{th} activity will equal the marginal cost

of that activity. In the context of this model, the marginal cost of the j^{th} activity is a marginal opportunity cost. In this sense the marginal opportunity cost represents the incremental productivity foregone for an incremental change in the j^{th} activity operating all the remaining J-1 activities at their present levels. This incremental change in the j^{th} activity is represented by its weighted marginal contribution to resource utilization and by its weighted marginal contribution to output production. In each case, the weights reflect an amount of resource underemployment or production goal overachievement which conveys a sense of inefficiency. Inefficiency in this model is intended in the narrow sense of failure to achieve the i^{th} production goal or to utilize the i^{th} resource level exactly as indicated by the decision maker's i^{th} subjective weighting factor.

As the value of r is sequentially reduced at each iteration of the Created Response Surface Technique, r_ℓ characterizes this specific weighting of productivity relative to its associated opportunity cost. For each value of r, r_ℓ, a point on a productivity-opportunity cost hypersurface is generated by the technique. As the number of iterations increase beyond bound causing the value of r to approach zero, the entire hypersurface is generated. Each point on that surface represents the decision maker's subjective trade off between productivity and its associated opportunity cost as reflected by the different weighting factors, r_ℓ, as ℓ increases beyond all bound.

C. DUAL FORMULATION

1. Practical Application

The computational convenience and economic interpretation of the dual formulation of linear programs provides an alternative approach to the solution or analysis of a primal linear model. In a similar fashion the dual formulation of a primal nonlinear program may exhibit the computational and theoretical relationships of primal-dual linear programming theory.

Consider the dual formulation of (A):

$$\text{(D)} \quad \begin{aligned} &\text{MIN:} \quad G(\bar{x},\bar{\lambda}) = f(\bar{x}) + \sum_{i=1}^{I} \lambda_i (g_i(\bar{x}) - k_i) \\ &\text{S.T.} \quad \frac{\partial G(\bar{x},\bar{\lambda})}{\partial x_j} = 0 \qquad j = 1,\ldots,J \\ &\qquad\quad \bar{\lambda} \geq 0 \quad \bar{x} \geq 0 \end{aligned}$$

The elements of the dual problem are:

1. Dual objective function: $G(\bar{x},\bar{\lambda})$
2. Primal objective function: $f(\bar{x})$
3. i^{th} deviation: $(g_i(\bar{x}) - k_i)$
4. i^{th} Lagrangian multiplier: λ_i
5. Aggregate deviation: $\sum_{i=1}^{I} \lambda_i (g_i(\bar{x}) - k_i)$

The minimization of the dual objective function subject to the specified necessary conditions should decrease the value of the aggregate deviation relative to the value of

the primal objective function. At optimality, the value of the dual objective function should identically equal the value of the primal objective function.

Generally, the solution of (D) may not provide the computational convenience demonstrated in the linear primal-dual relationship. The convexity assumptions of the primal nonlinear problem are generally not present in the dual formulation. The constrained optimization of the dual objective function is itself a class of nonlinear programs for which few solution techniques are available.

2. <u>Theoretical Application</u>

The dual formulation (D) may be solved implicitly by utilizing the established primal-dual feasibility property of the Created Response Surface Technique. The value of λ_i^* is determined by equating the necessary condition for (B) and (C) for a fixed value of r, r_ℓ.

$$\lambda_i^*(r_\ell) = \frac{r_\ell w_i}{(g_i(\overline{x})-k_i)^2} \quad .$$

The optimum value of $\overline{x}$, $\overline{x}_\ell^*$, for each created response surface generated by ℓ^{th} value of r satisfies the necessary condition of (B). Each $(\overline{x}_\ell^*, r_\ell)$ generates a dual multiplier $\overline{\lambda}(r_\ell^*)$ such that $(\overline{x}_\ell^*, \overline{\lambda}(r_\ell^*))$ is dual feasible satisfying the dual necessary conditions and providing an upper bound for the optimal value of (A) at the ℓ^{th} iteration.[9] As ℓ increases beyond bound then

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[9] Fiacco and McCormick, op. cit., p. 345.

$$P(\overline{x}^*_\ell, r_\ell) = G(\overline{x}^*_\ell, \overline{\lambda}(r^*_\ell)) = f(\overline{x}^*_\ell).$$

In the classical Lagrangian theory, the multiplier is customarily interpreted as a "shadow price" or "inputed value". At optimality the value of this multiplier is a representation of the opportunity cost of a change in the primal objective function in terms of a unit change in production or resource level.

Similarly, $\lambda^*_i(r_\ell)$ may be interpreted as a shadow price. At the optimum value of the Created Response Function for the ℓ^{th} iteration, the value of $\lambda^*_i(r_\ell)$ represents the amount of change in the Created Response Function for a unit change in the i^{th} goal deviation for $i = 1,\ldots,k$ or in the i^{th} resource usage for $i = k+1,\ldots,I$.

As indicated previously, iterative solutions to (B) for each value of r yield the following multiplier values in this problem.

$$\lambda^*_i(r_\ell) = \frac{r_\ell w_i}{(g_i(\overline{x})-k_i)^2} \qquad i = 1,\ldots,k$$

$$\lambda^*_i(r_\ell) = \frac{r_\ell w_i}{(k_i-g_i(x))^2} \qquad i = k+1,\ldots,I.$$

As an example of the theoretical interpretation of the dual formulation of a nonlinear program, consider the dual problem to (C).

$$\text{MIN:} \quad G(\overline{x},\overline{\lambda},\overline{\mu}) = f(\overline{x}) + \sum_{i=1}^{k} \lambda_i(g_i(\overline{x})-k_i) + \sum_{i=k+1}^{I} \quad \cdot$$

(E)

(E) Cont'd.

$$\lambda_i(k_i - g_i(\bar{x})) + \sum_{j=1}^{J} \mu_j x_j$$

$$\text{S.T.} \quad \frac{\partial G(\bar{x},\bar{\lambda},\bar{\mu})}{\partial x_j} = 0 \qquad \begin{array}{ll} i = 1,\ldots,k & \text{(output goals)} \\ i = k+1,\ldots,I & \text{(input levels)} \end{array}$$

$$\bar{\lambda} \geq 0 \ , \ \bar{\mu} \geq 0 \qquad j = 1,\ldots,J.$$

Solving the necessary conditions of (E) and substituting the value of μ_j into (E) yields the following modification to the problem.

$$\text{MIN: } G(\bar{x},\bar{\lambda}) = f(\bar{x}) + \sum_{i=1}^{k} \lambda_i(g_i(\bar{x})-k_i) + \sum_{i=k+1}^{I} \lambda_i(k_i-g_i(\bar{x}))$$

$$+ \sum_{j=1}^{J} x_j \left\{ \left[\sum_{i=1}^{k} \lambda_i \frac{\partial g_i(\bar{x})}{\partial x_j} - \sum_{i=k+1}^{I} \lambda_i \frac{\partial g_i(\bar{x})}{\partial x_j} \right] - \frac{\partial f(\bar{x})}{\partial x_j} \right\}$$

$$\text{S.T.} \quad \frac{\partial G(\bar{x},\bar{\lambda})}{\partial x_j} = 0$$

$$\lambda \geq 0.$$

Intuitively, minimization of the dual objective function should cause a reduction in the value of each term in the objective function. Because the minimization of the primal objective function is not economically interpreted in the context of this problem, assume that this term has reached its optimal value and may be considered fixed.

The second and third term of the dual objective function represents the total imputed value of the k goal deviations and the total imputed value of the I-k resource deviations respectively. The minimization process seeks to reduce the value of both terms. At optimality both terms vanish. Viewed in a complementary slackness context, either of the following groups of conditions may exist for each of the k goal deviations for the I-k resource deviations:

1. If $\left\{\begin{array}{ll} g_i(\bar{x})-k_i > 0 & i = 1,\ldots,k \\ k_i-g_i(\bar{x}) > 0 & i = k+1,\ldots,I \end{array}\right. \Rightarrow \lambda_i = 0$

2. If $\left\{ \lambda_i > 0 \quad i = 1,\ldots,I \right. \Rightarrow g_i(\bar{x}) = k_i \quad i = 1,\ldots,I$

The third term of the dual objective function is a composite term of marginal productivity and marginal opportunity cost. The interpretation of the elements of this term are the following:

1. $\dfrac{\partial f(\bar{x})}{\partial x_j}$ represents the internal marginal index of productivity. The index of productivity is defined by $f(\bar{x})$ which is a function of the activity levels.

2. $\sum_{i=1}^{k} \lambda_i \dfrac{\partial g_i(\bar{x})}{\partial x_j}$ represents the total, imputed, marginal value of output technology for the ℓ^{th} iteration of the Created Response Surface Technique procedure.

3. $\sum_{i=k+1}^{I} \lambda_i \dfrac{\partial g_i(\bar{x})}{\partial x_j}$ represents the total, imputed, marginal value of input technology for the ℓ^{th} iteration of the Created Response Surface Technique procedure.

The difference of (2) and (3) represents the imputed marginal opportunity cost of the j^{th} activity level at the ℓ^{th} iteration of the Created Response Surface Technique procedure. Thus, the entire term represents the total, weighted (by the level of the j^{th} activity), marginal, imputed benefit of operating the j^{th} activity.

Invoking the complimentary slackness conditions, the following conclusions result at optimality:

(1) If $x_j > 0$

then

$$\sum_{i=1}^{k} \lambda_i \frac{\partial g_i(\bar{x})}{\partial x_j} - \sum_{i=k+1}^{I} \lambda_i \frac{\partial g_i(\bar{x})}{\partial x_j} = \frac{\partial f(\bar{x})}{\partial x_j} .$$

If the j^{th} activity is utilized at a positive level, the marginal imputed opportunity cost of operating the j^{th} activity must equal the internal marginal index of productivity of operating the j^{th} activity.

(2) If

$$\sum_{i=1}^{k} \lambda_i \frac{\partial g_i(\bar{x})}{\partial x_j} - \sum_{i=k+1}^{I} \lambda_i \frac{\partial g_i(\bar{x})}{\partial x_j} > \frac{\partial f(\bar{x})}{\partial x_j}$$

then

$$x_j = 0.$$

If the marginal imputed opportunity cost of operating the j^{th} activity is strictly greater than the internal marginal index of productivity, then the activity should not be utilized at a positive level.

IV. DECENTRALIZED PLANNING MODELS

A. INTRODUCTION

In contemporary organization theory, organizations within an economy are characterized by a limited capacity to accumulate and process technical production information at a central level. Embedded in an economy of complex technologies of production, organizations are structured to provide the appropriate levels of specialization and division of labor to generate information and to utilize production processes to achieve specified organizational goals. Organizational goals are of two types: internal and external. Internal goals are those objectives motivating departmental agencies. External goals are the organization's objectives within an economy.

Assume that an hypothetical organization has defined at least one external goal. Further assume that this organization is motivated to pursue the achievement of this goal within the conditions imposed on organizational behavior by the economy. In this thesis an organization will be considered decentralized if the coordinated achievement of internal, departmental goals specified by the organization's central agency will achieve the external organizational goals. It is assumed that this coordination is achieved by a system of incentives, rewards, and penalties for individual and group behavior and by some goal defining process which insures that internal goals are not unintentionally duplicated between departments.

In neoclassical theory the organization is modeled as if the internal allocation of the organization's resources among departmental agencies takes place in a perfectly competitive internal economy. In this context internal goals may be constrained output maximization or constrained cost minimization. The departmental agencies act as independent economic agents who purchase factors of production and provide the products which jointly achieve the organization's external goal in an external economy. The central agency acts as a consumer of departmental products and a producer of departmental factors of production.

As in the recontracting model of static market equilibrium, a competitive equilibrium is achieved in the factor and commodity markets of the internal economy. The necessary conditions which characterize this equilibrium in the commodity market are that the central agency's rate of commodity substitution equals the rate of product transformation for each department. Likewise, in the factor market the rates of commodity substitution for each of the J departments equals the rate of product transformation of the central agency. At optimality, the decentralization is achieved by the price system which equates the quantities of departmental outputs demanded by the central agency to the quantities of outputs supplied by the J departments.

Recall that externalities, which are commodities for which markets do not exist, are not present in a perfectly competitive economy. In the absence of externalities, the

price system provides the necessary information to decentralize the organization.

B. LINEAR MODEL WITHOUT EXTERNALITIES

Several mathematical programming techniques have been proposed to model the decentralization of an organization. One model proposed by Dantzig-Wolfe [9] is representative of this general class of decomposition algorithms. This model will be presented to introduce the basic concepts of decomposition and to demonstrate how behavioral models of the agencies within an organization interact to achieve optimal behavior by a sequential decentralized planning procedure.

Assume that a hypothetical organization consists of J departments each with a defined, independent, internal goal. The J departments are interrelated by their common utilization of resources which are allocated to each of the J departments in a perfectly competitive internal economy. Further assume that the organization has at least one defined external goal and desires to pursue this goal in the external economy.

Consider the following model of this hypothetical organization.

$$\text{MAX:} \quad \sum_{j=1}^{J} \bar{p}_j' \bar{x}_j$$

$$\text{S.T.} \quad \sum_{j=1}^{J} \bar{A}_j \bar{x}_j = \bar{b}_o$$

$$\bar{B}_j \bar{x}_j = \bar{b}_j \qquad j = 1,\ldots,J$$

$$x_j \geq 0$$

where

$\bar{p}_j$ is a vector $(N_j x1)$ of market prices pertaining to the j^{th} department.

$\bar{x}_j$ is a vector $(N_j x1)$ of the j^{th} department's activity levels.

$\bar{A}_j$ is a matrix $(M_o x N_j)$ which describes how each of the j departments interact to utilize the organization resources.

$\bar{b}_o$ is a vector $(M_o x1)$ of resources under the control of the control agency.

$\bar{B}_j$ is a matrix $(M_o x N_j)$ of the independent constraints under the control of the j^{th} department.

$\bar{b}_j$ is a vector $(M_j x1)$ of resources under the control of the j^{th} department.

Assume that $\bar{B}_j \bar{x}_j = \bar{b}_j$ defines a closed and convex set for each $j = 1,\ldots,J$ such that $\bar{x}_j$, a feasible point of the j^{th} region, may be written as a convex combination of the extreme points, $\bar{x}^*_{jk}$, $k = 1,\ldots,Kj$ of that region.

$$\bar{x}_j = \sum_{k=1}^{Kj} \bar{\rho}_{kj} \, \bar{x}^*_{kj}$$

$$\sum_{k=1}^{Kj} \rho_{kj} = 1$$

Substituting the above conditions into the original problem produces the "full master program".

$$\text{MAX:} \quad \sum_{j=1}^{J} \sum_{k=1}^{Kj} \bar{p}_j \bar{\rho}_{kj} \bar{x}^*_{kj}$$

$$\text{S.T.} \quad \sum_{j=1}^{J} \sum_{k=1}^{Kj} \bar{\rho}_{kj} \bar{A}_j \bar{x}^*_{kj} = \bar{b}_o$$

$$\sum_{k=1}^{Kj} \rho_{kj} = 1$$

$$\bar{\rho}_{kj} \geq 0 \qquad j = 1,\ldots,J$$

$$k = 1,\ldots,Kj$$

The full master program has fewer constraints than the original program, but it does have more variables. The variables of the full master program will be generated only as needed in the solution procedure.

Let $\bar{B}^{(t)}$ denote a basis to the full master program at iteration t such that $\rho_B^{(t)}$, a vector of basic variables, may be written as:

$$\rho_B^{(t)} = \bar{B}^{(t)-1}\, \bar{b} \qquad t = 1,\ldots,T$$

where

$$\bar{b} = \begin{bmatrix} \bar{b}_o \\ \hline 1 \end{bmatrix}.$$

Then the dual variables to the full master program may be written as:

$$\bar{\sigma}^{(t)} = [\bar{\sigma}_1^{(t)}, \bar{\sigma}_2^{(t)}] = p_B\, \bar{B}^{(t)-1}$$

where

$\bar{p}_B$ is a vector ($1xM_o+J$) of prices associated with the basic variables

$\bar{\sigma}_1^{(t)}$ is a vector ($1xM_o$)

$\bar{\sigma}_2^{(t)}$ is a vector ($1xJ$).

$\bar{\sigma}_1^{(t)}$ is the central agency's imputed value of an additional unit of each of the M_o resources. This vector is generated from a basis consisting of vectors which represent how some or all of the J departments are currently utilizing an allocation of these common resources. This utilization is imbedded in $\bar{\sigma}_1^{(t)}$.

$\bar{\sigma}_2^{(t)}$ is the central agencies imputed value of the past proposals of resource utilization from each of the J departments as reflected in the current basis, $\bar{B}^{(t)}$.

$\bar{\sigma}^{(t)}$ is used as a prospective index at the t^{th} iteration of the procedure. It represents the central agency's request for production information.

After receiving this prospective index, each of the J departments determine if their previous proposal of resource utilization, their production plan, was optimal for the central agency. This is accomplished by computing the following criteria function with the information provided in the prospective index,

$$(\bar{\sigma}_1^{(t)}\bar{A}_j - \bar{p}_j)\ \bar{x}_j^{(t-1)} + \bar{\sigma}_{2j}^{(t)} \qquad \begin{array}{l} t = 1,\ldots,T \\ j = 1,\ldots,J \end{array}$$

where,

$\bar{x}_j^{(t-1)}$ is the previous production plan of the j^{th} department in the form of activity levels.

$\bar{\sigma}_1^{(t)}\bar{A}_j\bar{x}_j^{(t-1)}$ is the central agency's imputed cost of the j^{th} department's resource utilization for the previous production plan.

$\bar{p}_j\bar{x}_j^{(t-1)}$ is the market value of the j^{th} department's production plan.

σ_{2j} is the net profitability of all production plans of the j^{th} department before the previous plan.

Thus criterion function is an expression of the net profitability of the previous proposal of each of the J departments.

If the j^{th} criterion function is nonnegative, this indicates that the j^{th} department is utilizing resources optimally reflecting a nonnegative economic profit. If the minimum value of all J criterion functions is nonnegative, this indicates that all departments are utilizing resources optimally implying that the optimal resource allocation plan for the organization has been achieved.

If the criterion function is negative for any of the J departments, this indicates that these departments are not utilizing resources properly and should "improve" their production plans by providing a new proposal.

Assume that one criterion function is negative for some $j = e$ at the t^{th} iteration. This indicates that the e^{th}

department is not utilizing resources in an optimal manner for the organization. The e^{th} department generates a new production plan $\bar{x}_e^{(t)}$ by solving the e^{th} departmental program.

$$\text{MIN:} \quad (\bar{\sigma}_1^{(t)} \bar{A}_e - \bar{p}_e) \bar{x}_e^{(t)} + \sigma_{2e}^{(t)}$$

$$\text{S.T.} \quad \bar{B}_e \bar{x}_e^{(t)} = \bar{b}_e$$

$$\bar{x}_e^{(t)} \geq 0.$$

Utilizing the standard simplex criterion, a new basis, $\bar{B}^{(t+1)}$, is formed in the full master program by the replacement of an existing basis vector with $\bar{q}_e^{(t)}$, a vector constructed from $\bar{x}_c^{(t)}$, suitable for the dimension of the master program. This new basis provides $\bar{\rho}_B^{(t+1)}$ and $\bar{\sigma}^{(t+1)}$.

At each iteration, t = 1,...,T, each department recalculates its criterion function. This recalculation by all departments serves to indicate which departments have achieved optimal production plans at this iteration and to provide a check on previously optimal plans which may be adversly effected by subsequent reallocations of resources to other nonoptimal departments.

If more than one criterion function is negative, the department with the most negative criterion function should generate a new proposal.

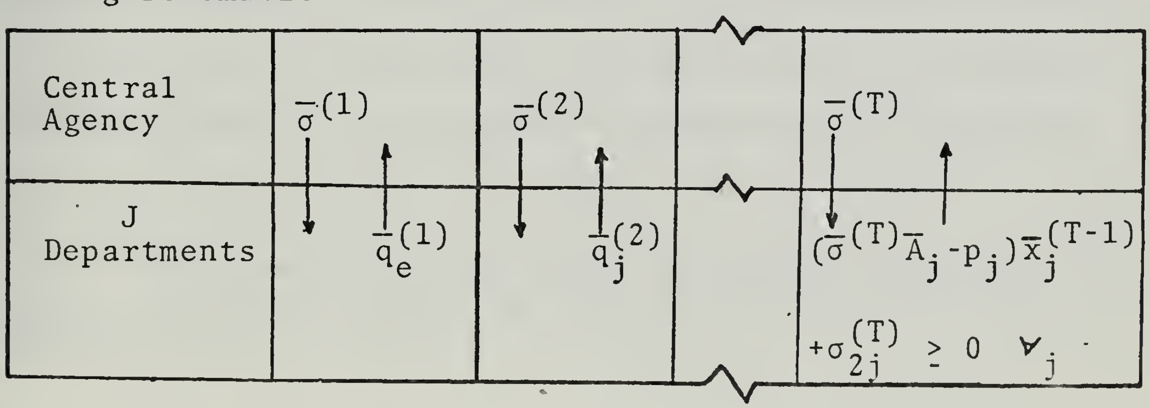
At each iteration of the procedure, only one department is allowed to return a revised production plan. Each revised

production plan generates a new master basis initiating another optimality evaluation by all departments. The iterations continue until the optimal solution to the full master program is achieved. The optimal solution to the full master program consists of the appropriate activity levels for all J departments.

It should be noted that the optimal solution to the full master program may contain more than one proposal from some of the departments since this optimal solution consists of M_o+J basic vectors of which there can exist J unique individual proposals at most. Consistent with the concept of decentralization presented in this thesis it is assumed that if more than one proposal is optimal for any department then the central agency will specify the sequence of fulfilling these proposals.

This decomposition procedure may be initiated by utilizing existing operating levels as a basis. If no such basis exists, the artificial basis method of the revised simplex algorithm may be used.

The iterative procedure may be described in the following schematic:



where,

$\bar{\sigma}^{(t)}$ is the prospective index at the t^{th} iteration.

$\bar{q}_e^{(t)}$ is the e^{th} departmental revised production plan.

Nonlinearities in either the objective function or in the constraints may be solved by the linear decomposition procedure as long as the objective function is concave or the constraints are convex functions. However, the termination of the procedure in a finite number of iterations can no longer be assured.[10] An appropriate stopping rule must be established to terminate the procedure.

Recently, several new algorithms may have been proposed for the solution of decomposable problems of the form with a concave nonlinear objective function and linear separable constraints. These algorithms employ gradient search methods to achieve local optimum.[11]

C. LINEAR MODEL WITH EXTERNALITIES

In the previous application of the Dantzig-Wolfe decomposition procedure, an organization was modeled as if it allocated its resources in an internal, perfectly competitive economy. In the absence of externalities within this internal economy, the perfectly competitive market mechanism achieved an optimal organization plan through

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[10] Huysmans, J., Some Notes on the Decomposition Principle, Externalities and Decentralization in the Firm, Working Paper, No. 141, p. 13, 1965.

[11] Fletcher, R., ed., "Large Step Gradient Methods for Decomposable Nonlinear Programming Problems" by L. Schwartz, Optimization, p. 106, Academic Press, 1969.

the imputed price system. This neoclassical model presumes that departmental behavior is competitive in the sense that each department acted out of "greed" to independently bid resources away from other departments.

Viewed in contemporary organization theory, departments are assumed to be psychically and, on occasion, physically interdependent. Psychic interdependence is caused by an overall spirit of departmental cooperation in achieving the organization's objective beyond personal and group rewards, incentives, or penalties. Physical interdependence may occur if one or more departmental goals are dependent upon other departmental goals. Both types of interdependencies would be considered externalities in the neoclassical model. No market exists to evaluate the effects of externalities on the achievement of an optimal organizational plan.

In contemporary organization theory, the presence of externalities in the organization model requires some additional form of information other than that provided in the internal price system.[12] Thus, additional information should serve to account for the beneficial or detrimental contributions of externalities in pursuing an optimal plan. Malinvaud suggests that production goals may be used as additional information in conjunction with the internal

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[12] Charnes, A., Clower, R., and Kortaneck, K., "Effective Control Through Coherent Decentralization with Preemptive Goals," Econometrica, Vol. 35, No. 2, p. 305, 1967.

price system.[13] This approach would approximate current procedures used in business organizations and provide possible application and validation of these models. This model insures that optimal achievement of departmental goals will result in the optimal achievement of the organizational goal.

A brief discussion of the model is presented to introduce the concept of goal decomposition and to provide an intuitive grasp of the interaction of goal specifications and imputed prices to achieve organization decomposition. Rigorous mathematical proof of the characteristics of the model may be found [20].

Consider the following model of an organization consisting of J interdependent departments.

$$\text{MIN:} \quad \sum_{j=1}^{J} \bar{c}_j' \bar{u}_j$$

$$\text{S.T.} \quad \sum_{j=1}^{J} \bar{C}_j \bar{u}_j \geq \bar{d}$$

$$\bar{B}_j \bar{u}_j \geq b_j \qquad j = 1,\ldots,J$$

$$\bar{u}_j \geq 0$$

[13] Malinvaud, E. and Barharach, M. O. L., eds., "Decentralized Procedures for Planning" by E. Malinvaud, Activity Analysis in the Theory of Growth and Planning: Proceedings of a Conference held by the International Economic Association, p. 208, St. Martin's Press, 1967.

where,

$\overline{c}_j$ is a vector (Mx1) of fixed costs for the unit level of the j^{th} departments activity.

$\overline{u}_j$ is a vector (Mx1) of the j^{th} department's activity levels.

$\overline{C}_j$ is a matrix (MxJ) which describes how each department's production interacts to achieve the organization's goals.

$\overline{d}$ is a vector (Jx1) of the central agency's external production goals.

$\overline{B}_j$ is a matrix (MxJ) of the independent constraints under the control of the j^{th} department.

$\overline{b}_j$ is a vector (Jx1) of the j^{th} departments minimum production goals.

The central agency's problem is to minimize total production cost subject to achieving minimal specified levels of departmental and organizational production.

Now consider the dual formulation of this model at the t^{th} iteration of the procedure.

$$\text{MAX: } \sum_{j=1}^{J} \overline{b}_j' \overline{x}_j^{(t)} + \overline{d}' \ \overline{\lambda}^{(t)} \qquad t = 1,\ldots,T$$

$$j = 1,\ldots,J$$

$$\text{S.T. } \overline{B}_j \overline{x}_j^{(t)} + \overline{C}_j \overline{\lambda}^{(t)} \geq \overline{c}_j$$

$$\overline{x}_j^{(t)} \geq 0 \quad \overline{\lambda}^{(t)} \geq 0$$

where,

$\bar{x}_j^{(t)}$ is a vector (Jx1) of the central agency's imputed value of one additional unit of production in the j^{th} department at the t^{th} iteration.

$\bar{\lambda}^{(t)}$ is the central agency's vector (Jx1) of imputed value of one additional unit of organizational production at the t^{th} iteration.

The problem is to maximize the aggregate imputed value of production subject to the requirement that aggregate imputed marginal benefit of an additional unit of production must at least equal the marginal cost of that unit of production at each iteration.

At optimality the following equality exists:

$$\sum_{j=1}^{J} \bar{c}_j' \bar{u}_j^{(t)*} = \sum_{j=1}^{J} \bar{b}_j \bar{x}_j^{(t)*} + \bar{d}\ \bar{\lambda}^{(t)*}$$

Let $\bar{\alpha}_j^{(t)*} = \bar{C}_j \bar{u}_j^{(t)*}$. Where $\bar{\alpha}_j^{(t)*}$ is the vector of optimal organizational production goals measured in value units at the t^{th} iteration.

Provided with an optimal internal pricing system, $\bar{\lambda}^{(t)*}$, at the t^{th} iteration, the j^{th} departmental problem is:

$$\text{MIN:}\quad u_j^{(t)'}\ (c_j - C_j \lambda^{(t)*})$$

$$\text{S.T.}\quad B_j u_j^{(t)} \geq b_j$$

$$u_j^{(t)} \geq 0.$$

As mentioned previously, the price system, $\bar{\lambda}^{(t)*}$, and additional information in the form of preemptive goals, $\bar{\alpha}_j^{(t)*}$,

are provided for each department at each iteration. In this context, a goal may be a vector or a scalar quantity which relates the activity levels of the previous proposals to the organizational production goals by some equational representation. In this case, the equational representation reflects the difference, if any, between the previous departmental proposals and the organizational objectives.

The original j^{th} departmental problem may be written:

$$\text{MIN:} \quad \bar{u}_j^{(t)\prime} (\bar{c}_j - \bar{C}_j \bar{\lambda}^{(t)*}) + M \,||\, \bar{\alpha}_j^{(t)*} - \bar{u}_j^{(t)\prime} \bar{C}_j \,||$$

$$\text{S.T.} \quad \bar{B}_j \bar{u}_j^{(t)} \geq \bar{b}_j$$

$$\bar{u}_j^{(t)} \geq 0$$

where "$||\quad||$" means the sum of or the MAX of

$$\{|\bar{\alpha}_j^{(t)*} - \bar{u}_j \bar{C}_j| \quad j\}.$$

M is a positive real number representing an arbitrarily large penalty cost for deviation from the value of the production goals.

Intuitively, the minimization of the objective function should cause a decrease in the value of the second term. This may be interpreted as minimizing absolute deviations from organizational production goals. Minimization of the first term, which may be rewritten as follows:

$$- \; (\bar{C}_j \bar{\lambda}^{(t)*} - \bar{u}_j^{(t)\prime} \bar{c}_j)$$

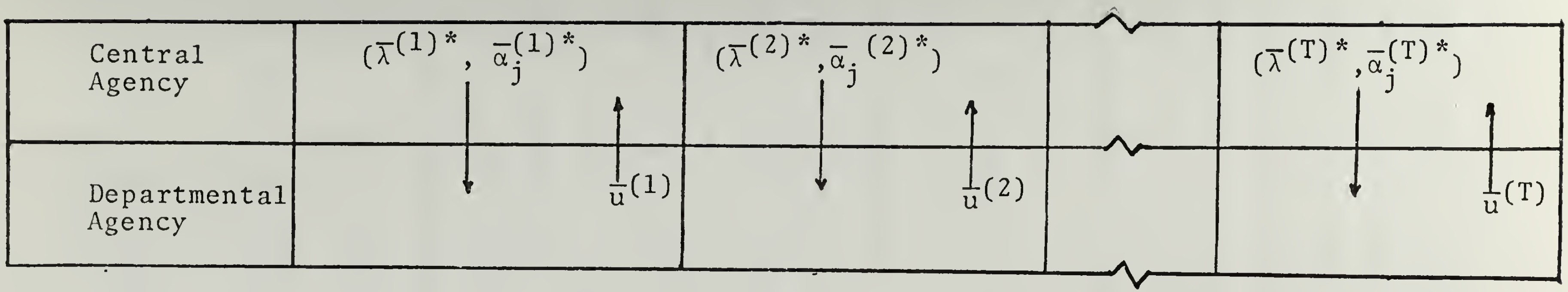
should cause this term to increase in absolute value. This indicates that the optimal imputed value of production levels should be at least as great as the cost of that level of production at each iteration.

Assume that the central agency has established a goal deviation tolerance as an optimality criterion. The procedure is initiated by generating $\bar{\alpha}_j^{(1)*}$ for $j = 1,\ldots,J$ and $\bar{\lambda}^{(1)*}$ by the organization from an existing or artificial basis of activity variables. Each department generates a new vector of activity levels from the supplied information. In return, revised activity levels, $\bar{u}_j^{(1)}$, for $j = 1,\ldots,J$, provide a new basis for reiteration of the central agency's program producing a revised prospective index, $(\bar{\lambda}^{(2)*}, \bar{\alpha}_j^{(2)*})$. Subsequent exchanges of prospective indices and proposals continue until an organizational optimum defined by the specified deviation tolerance is achieved. This iterative procedure is described in Figure 1.

D. LINEAR GOAL DECOMPOSITION MODEL

The Dantzig-Wolfe and preemptive goals decentralization models were independent of the formal hierachical structure of an organization. In these models, organizations were structured on two levels, a planning level and an operations level.

In contemporary theory, organization structure reflects the interrelationships among subordinate agencies created



where,

$$\bar{u}^{(t)} = \begin{bmatrix} \bar{u}_1^{(t)} \\ \cdot \\ \cdot \\ \cdot \\ \bar{u}_j^{(t)} \\ \cdot \\ \cdot \\ \cdot \\ \bar{u}_J^{(t)} \end{bmatrix}$$

Figure 1. Iterative Procedure Schematic.

within the organization to accomplish defined objectives. Models which consider organizational structure provide a means of including an environment in the analysis of the decision processes within the organization. Structure dependent models should implicitly reflect the effects of a particular structure in a decision process. It is not clearly evident whether the organizational structure can be separated from the decision process for analysis or optimization.

The Generalized Goal Decomposition model proposed by Ruefli [12] is a modified form of goal programming model of an organization which does not exhibit departmental or divisional externalities. The model is structure dependent and goal oriented. A three level model will be discussed, but the procedure is applicable to any finite number of organizational levels.

Consider an organization consisting of a central agency and J departments. Each department consists of E_j, e = 1,...,E_j, divisions.

The mathematical programming model of the organization's central agency is as follows:

$$\text{MAX:} \quad \sum_{j=1}^{J} \bar{\pi}_j^{(t)} \bar{g}_j^{(t)} \qquad t = 1,\ldots,T$$

$$j = 1,\ldots,J$$

$$\text{S.T.} \quad \sum_{j=1}^{J} \bar{P}_j \bar{g}_j^{(t)} \leq \bar{g}_o$$

$$\bar{g}_j^{(t)} \geq 0$$

where,

$\bar{\pi}_j^{(t)}$ is a vector $(1xM_j)$, $m = 1,\ldots,M_j$, of the imputed value of one unit of positive or negative derivation from the m^{th} goal to the j^{th} department at the t^{th} iteration.

$\bar{g}_j^{(t)}$ is a vector $(M_j x1)$ of the goal levels assigned to the j^{th} department.

$\bar{P}_j$ is a matrix $(M_j x M_j)$ of the J departments' joint utilization of organization resources.

$\bar{g}_o$ is a vector (Kx1), $K = \sum_{j=1}^{J} M_j$ of external constraint levels.

At each t iteration, the central agency's problem is to maximize the imputed value of aggregate departmental output subject to some resource constraint.

The mathematical programming model of the j^{th} department is as follows:

$$\text{MIN:} \quad \bar{w}_j^{(+)}\bar{y}_j^{(+)} - \bar{w}_j^{(-)}\bar{y}_j^{(-)} \qquad t = 1,\ldots,T$$

$$j = 1,\ldots,J$$

$$\text{S.T.} \quad \sum_{e=1}^{E_j} \bar{a}_{e_j}^{(t)} \, x_{e_j} - I_{m_j} \, \bar{y}^{(+)} + I_{m_j} \, \bar{y}^{(-)} = \bar{g}_j^{(t)}$$

$$0 \leq x_{e_j} \leq 1$$

$$\bar{y}_j^{(+)}, \bar{y}^{(-)} \geq 0$$

where,

$\bar{w}_j^{(+)}, \bar{w}_j^{(-)}$ are vectors ($1xM_j$) of *a priori* weights for positive or negative deviations from goals.

$\bar{y}_j^{(+)}, \bar{y}_j^{(-)}$ are vectors (M_jx1) of positive or negative deviations from the M_j goals of vector $\bar{g}_j^{(t)}$ for the j^{th} department.

$\bar{a}_{e_j}^{(t)}$ is a vector (M_jx1) of the E_j divisions joint achievement of departmental goals at the e^{th} iteration.

x_{e_j} is a scalar level of activity of the e^{th} division.

I_{m_j} is an identity martrix (M_jxM_j).

At each t iteration, the j^{th} department's problem is to minimize it's aggregate weighted goal deviations subject to the technology of achieving the department's goal. This technology describes how the E_j divisions interact to achieve departmental goals.

The mathematical programming model of the e^{th} division is as follows:

$$\text{MIN:} \quad \bar{\pi}_j^{(t)} \bar{a}_{e_j}^{(t)} \qquad t = 1,\ldots,T$$

$$\qquad\qquad\qquad\qquad e = 1,\ldots,E_j$$

$$\text{S.T.} \quad \bar{D}_{e_j} \bar{a}_{e_j}^{(t)} \geq \bar{f}_{e_j}$$

$$\bar{a}_{e_j}^{(t)} \geq 0$$

where,

$\overline{D}_{e_j}$ is a matrix $(M_j x M_j)$ of the e^{th} division's technology.

$\overline{a}^{(t)}_{e_j}$ is a vector $(M_j x1)$ of the variable activity levels for the e^{th} division.

$\overline{f}_{e_j}$ is a vector $(M_j x1)$ of minimum output levels.

At each iteration, the e^{th} division's problem is to minimize the imputed value of their production subject to the physical technology of that production.

The central agency initiates the procedure by providing J prospective indices, $\overline{g}^0_j$, for each of the j departments. Each index contains production or resource goals for the j^{th} department. The initial indices may reflect current operating conditions or mature jedgment in forecasting goal levels.

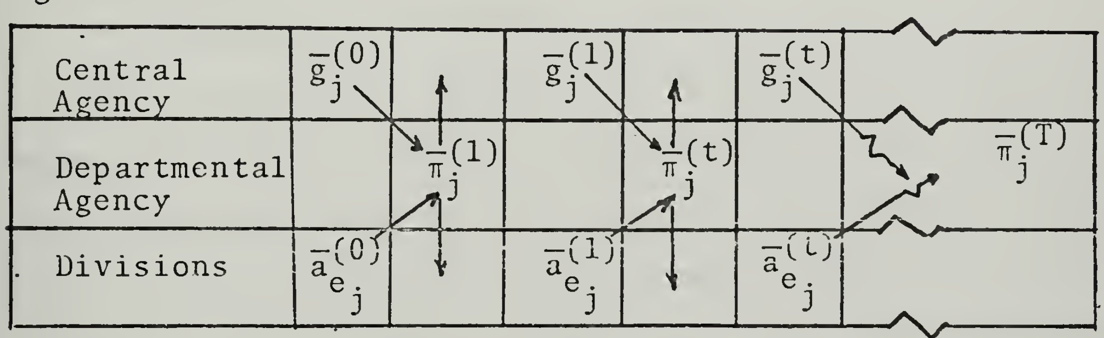
Each of the J departments, with a previous technology coefficient vector, $\overline{a}^0_{e_j}$, seeks to minimize the weighted deviations from their respective goals. At optimality, the dual variables to this program solution are the shadow prices, $\overline{\pi}^{(1)}_j$. The J departments respond to the central agency with a proposal of shadow prices. This vector of shadow prices is provided to each of the divisions of the J departments as a prospective index.

Having received a prospective index for this iteration, the e^{th} division seeks to minimize the imputed value of its production. The optimal solution of the e^{th} division program yields a proposal, $\overline{a}^{(1)}_{e_j}$, for it's department.

Concurrently, after receiving the proposals, $\bar{\pi}_j^{(1)}$, from each department, the central agency optimizes it's revised program providing new prospective indices, $\bar{g}_j^{(1)}$, to each department for the next iteration.

Provided with a new technology coefficient vector $\bar{a}_{e_j}^{(1)}$ by their respective divisions and new goal levels, $\bar{g}_j^{(1)}$ by the central agency, each of the j departments optimizes the revised program generating a new vector of shadow prices.

The iterative procedure may be described by the following schematic:



The procedure terminates when the deviations from the departments' goals are within prescribed tolerance limits or at a minimum value for which no readjustment of central agency's goals indices or the divisions proposals cause any change in the department's objective functions.

Because the organizational model consists of processes which are each finite and composable into one problem, the entire processes will reach an optimum in a finite number of iterations.[14]

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[14] Byrne, F. R. and others, eds., "PPBS-An Analytic Approach" by T. Ruefli, <u>Studies in Budgeting</u>, p. 173, North Holland Publishing Company, 1971.

If departmental or divisional externalities are relevant to the model of organization, the Centralized Goal Decomposition model cannot achieve decentralization by prices along. However, the preemptive goals procedure of decentralization proposals in [20] may be used if the requisite conditions of that model are met. The generalized goal decomposition model is readily adaptable to the preemptive goals procedure because the Centralized Goal Decomposition and the preemptive goals are both based on goal programming.

E. NONLINEAR GOAL DECOMPOSITION MODEL

As a conjectural variation of the generalized goal decomposition model, consider a hypothetical three level organization which consists of a central agency and J departments. Each department consists of E_j divisions which may exhibit production externalities within their respective departments but not between departments. In this model production externalities will occur when the output of one division shares an intentional or unintentional characteristic with the output of one other division. This condition exists when a lesser included output, a by-product, of one division is the same as the principal output of one other division of the same department. This by-product is not accounted for in the pseudo-market mechanism within the organization.

Recall that in a perfectly competitive market, this production externality would not be reflected in the pricing system. Likewise, in a decentralization model which relies on an internal price system alone to provide the information

necessary to achieve decentralization, this externality would not be reflected in that internal pricing system. If these externalities are not taken into account in the decentralization model, solutions generated by that model will be suboptimal.

Two possible ways of dealing with externalities are to merge the divisions which generate this common output or to restructure the model.

Divisional merger internalizes the by-product so that its effects on organizational goal achievement are reflected in the price system. Merger may be appropriate if the common output of related divisions is such that the organization views this combined production of a common output to be more important than the previous individual outputs. Merger may decrease organizational information transaction costs which are assumed to increase exponentially with organizational expansion.

Divisional merger may be inappropriate if the budgeting system or the time sequencing of related divisional activities requires that distinct divisional lines must be maintained. This condition may occur if the divisions represent fixed duration projects or fixed budgetary categories.

A restructuring of the model should account for the additional information required to reflect the beneficial or detrimental effects of divisional externalities. As indicated in the preemptive goals model, the internal price system of the Dantzig-Wolfe model was insufficient to decentralize the

organization with departmental externalities. The preemptive goals in value form provided this additional information to account for those externalities. The Dantzig-Wolfe model was restructured to accommodate this view of organizational behavior. This approach presupposed the ability of the central agency to interpret deviations from organizational objectives in terms of departmental goals.

As a different approach to this process of generating the supplementary information required for decentralization, consider the pairwise divisional externalities of this hypothetical organization as a type of departmental collective good. It's collective properties stem from the necessity of the department to consume the benefit or detriment of the interactive output of interdependent divisions without exclusion whenever both divisional activities operate at a nonzero level. In light of this collective property, it becomes incumbent upon the department to account for the effects of these interactions in the goal achievement effort. Under these conditions departments cannot be viewed as "deviation minimizers" as proposed in the pure Generalized Goal Decomposition model.

Consider the following mathematical programming formulation of the j^{th} departmental at the t^{th} iteration of the procedure.

$$\text{MAX: } \sum_{k=1}^{E_j} \sum_{e=1}^{E_j} b_{ke_j}^{(t)} x_{k_j} x_{e_j} + \sum_{i=1}^{E_j} b_{i_j}^{(t)} x_{i_j}^{(t)} \qquad \begin{array}{l} t = 1,\ldots,T \\ j = 1,\ldots,J \end{array}$$

$$\text{S.T.} \sum_{e=1}^{E_j} a_{ke_j}^{(t)} x_{k_j}^{(t)} x_{e_j}^{(t)} + \sum_{i=1}^{E_j} a_{i_j}^{(t)} x_{i_j}^{(t)} \geq \bar{g}_j^{(t)}$$

$$x_{k_j}^{(t)}, x_{e_j}^{(t)}, x_{i_j}^{(t)} \geq 0$$

where,

$b_{i_j}^{(t)}$, $b_{ke_j}^{(t)}$ are scalar measures of the present, certain, benefit or detriment derived from a unit level of activity under the following conditions:

$b_{i_j}^{(t)}$	> 0	is the i^{th} division's beneficial output as if the e^{th} division was was independent.
$b_{ke_j}^{(t)}$ is replaced by	$\frac{b_{ke_j}^{(t)}}{2}$	if $e \neq k$ and divisions e and k have a common beneficial output. This replacement precludes double counting of benefit in the objective function.
$b_{ke_j}^{(t)}$ is replaced by	$\frac{-b_{ke_j}^{(t)}}{2}$	if $e \neq k$ and divisions e and k have a common detrimental output. This replacement precludes double counting of detriment in the objective function.
$b_{ke_j}^{(t)}$	$= 0$	is $e \neq k$ and divisions e and k are independent.

It is assumed that $b_{i_j}^{(t)}$, $b_{ke_j}^{(t)}$ are commensurable for all e and k. This commensurability may be achieved by using a

suitable numeraire which generates an appropriate benefit accounting system for all the divisions' outputs. The objective function formed from this accounting system is assumed to be a concave function.[15]

$x_{e_j}^{(t)}$ is a scalar activity level of the e^{th} division "as if" it were independent of all other divisions within the j^{th} department.

$x_{k_j}^{(t)}$ is a scalar activity level of the k^{th} division "as if" it were independent of all other divisions within the j^{th} department.

$x_{e_j}^{(t)} x_{k_j}^{(t)}$ is a joint scalar activity level of the e^{th} and k^{th} divisions interdependent activity levels.

$a_{i_j}^{(t)}$, $a_{ke_j}^{(t)}$ are scalar technological coefficients under the following conditions:

$a_{i_j}^{(t)} \geq 0$ is the i^{th} division's technological coefficient as if the i^{th} division was independent.

$a_{ke_j}^{(t)} \geq 0$ if $e \neq k$ and divisions e and k have common output.

$a_{ke_j}^{(t)} = 0$ if $e \neq k$ and divisions e and k are independent.

$\bar{g}_j^{(t)}$ is a vector $(M_j x1)$ of the goal constraints assigned to the j^{th} department by the central agency.

The department's problem at each iteration is to maximize the total present, certain, benefit of its departmental activities subject to the technological interaction of achieving

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[15] The quadratic term will be concave if the quadratic form of that term is negative semi-definite.

specified goals. But it must solve this problem contingent upon the generation of the necessary information to structure the problem at each iteration.

As in the previous models of organizational decentralization, a planning and controlling agency requires specific information which it does not have the ability to accumulate and process. Likewise, in this model the departments which exhibit divisional production interactions require not only the technical information needed by all departments without divisional externalities but also they require additional information to achieve their departmental objectives. Thus, the informational requirement for these departments is of two types: technical and artificial.

In this model it is assumed that departments use prospective indices to request specific technological information from all their subordinate divisions. This technical information is implicitly embedded in each divisional proposal. Divisions are assumed to behave as if they were independent and ignorant of production externalities.

In this thesis, artificial information is a subjective evaluation of the amount of benefit or detriment contributed to departmental goal achievement by production externalities between two interdependent divisions. This benefit or detriment is reflected in the scalar value of:

$b_{i_j}^{(t)}$ for independent divisions.

$b_{ke_j}^{(t)}$ for $e \neq k$ and divisions e and k interdependent.

$a_{ke_j}^{(t)}$ for $e \neq k$ and divisions e and k interdependent.

It is assumed that this artificial information will be generated from divisional proposals by some form of logical analysis or study initiated and accomplished by each department as necessary.

Because of the nonlinearity of the objective function and the nonlinearity of constraint equations, the solution of the j^{th} departments problem in its present form cannot be achieved by linear or quadratic programming algorithms. Linear assumptions in the objective function or the constraint equations would eliminate the interactive effects of interdependent divisional activities.

Consider the Created Response Surface Technique formulation of the j^{th} departments program with some pairwise divisional interdependencies at the t^{th} iteration.

$$\text{MAX: } P(\bar{x}_m^j, r) = \sum_{k=1}^{E_j} \sum_{e=1}^{E_j} b_{ke_j} x_{k_j} x_{e_j} + \sum_{i=1}^{E_j} b_{i_j} x_{i_j}$$

$$- r \left\{ \frac{w_1}{g_{1_j} - a_{1_j} x_{1_j} - a_{12_j} x_{1_j} x_{2_j}} + \frac{w_2}{g_{2_j} - a_{2_j} x_{2_j} - a_{21_j} x_{1_j} x_{2_j}} \right.$$

$$\left. + \frac{w_3}{g_{3_j} - a_{3_j} x_{3_j}} + \cdots + \frac{w_{E_j}}{g_{E_j j} - a_{E_j j} x_{E_j j}} \right\}$$

$$- r \left\{ \underline{X} + \underline{Y} \right\}$$

where,

$\bar{x}_m^j$	is a vector $(M_j x1)$ of activity levels of the E_j divisions of the j^{th} department at the m^{th} iteration of the gradient search procedure of the response surface technique.
$\dfrac{w_1}{g_{1_j} - a_{1_j} x_{1_j} - a_{12_j} x_{1_j} x_{2_j}}$	is the weighted reciprocal deviation from the 1^{st} departmental goal consisting of:
g_{i_j}	the 1^{st} of M_j goals of the j^{th} department.
a_{1_j}	is the technological level proposed by the 1^{st} division of the j^{th} department.
x_{1_j}	is the variable activity level of the 1^{st} division.
a_{12_j}	is the interaction level proposed by the j^{th} department from received coefficients a_{1_j} and a_{2_j} from divisions 1 and 2 to account for operating divisions 1 and 2 jointly with some common output.
$x_{1_j} x_{2_j}$	variable joint activity level of divisions 1 and 2.

r	penalty term weighting factor which serves as a benefit-goal conversion factor in conjunction with each respective w_i individual weighting factor.
$\underline{X}$ and $\underline{Y}$	are the penalty term associated with the nonnegative constraints of the variables.

The central agency initiates the procedure by providing J prospective indices, $\bar{g}_j^{(0)}$, for each of the j departments. Each index contains production goals for the j^{th} department. The initial indices may reflect current operating conditions or mature judgment in forecasting goal levels.

Each of the j departments have available a previous technology coefficient vector $\bar{a}_{i_j}^{0}$ consisting of the proposed technological coefficients of each of the E_j divisions "as if" each division were independent of each other. The j departments determine among which of their respective divisions production externalities exist. Once the interdependencies are established, the technological level of interdependence is determined by the department. The technology coefficients from an upper triangular matrix whose diagonal elements are the divisions' coefficients "as if" they were independent, and the off diagonal elements are the positive coefficients indicating divisional pairwise interdependency or zero indicating divisional pairwise independence generated by the j^{th} department. Likewise, the appropriate benefit or detriment

coefficients for each unit of independent activity and joint activities are generated. The created response function is optimized until an established optimality criterion is achieved at this iteration. This particular vector of the variable activity levels and penalty term weighting factor generate a vector of dual variables, $\bar{\lambda}_j^{(1)}$, which serve as a shadow price. This shadow price vector becomes a prospective index to each of the E_j divisions of the j^{th} department and a proposal in response to the central agency's initial index. The shadow price vector has accounted for the divisional externalities.

The e^{th} division seeks to minimize the imputed value of its production. Embedded in this process is an accounting of the interaction of the e^{th} division with the k^{th} division. This interaction is reflected in the shadow price which was computed for this iteration by using technical and artificial information. The optimal solution of the i^{th} division program provides a proposal $a_{i_j}^{(1)}$ for its respective department.

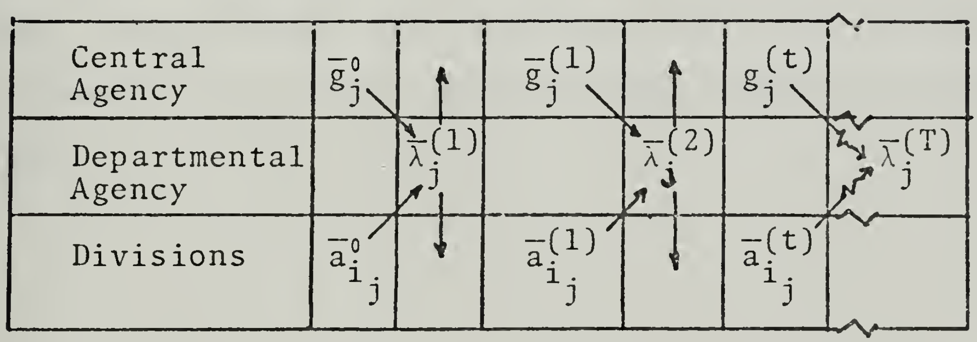
After receiving proposals, $\bar{\lambda}_j^{(1)}$, from each department, the central agency optimizes its revised program providing new prospective indices, $\bar{g}_j^{(1)}$, to each department.

Provided with a new technology coefficient vector $\bar{a}_{i_j}^{(1)}$ from their respective divisions and new goal levels, $\bar{g}_j^{(1)}$, each of the j departments determine which divisions are interdependent since at any iteration it may be optimal not to operate some activities at a positive level. Once these interdependencies are established, the technology coefficients

and benefit coefficients are generated or altered as appropriate. The created response function is optimized generating a new vector of shadow prices $\bar{\lambda}_j^{(2)}$ reflecting the revised goals, technological coefficients and effects of externalities.

The procedure terminates if the change in total benefit is within a prescribed tolerance limit or at a value for which no readjustment of central agency prospective indices or divisions proposals causes any significant (a departmental criterion) change in the department's objective function.

This iterative procedure may be described in the following schematic:



V. SUMMARY

In this thesis the Created Response Surface Technique of solving a class of nonlinear mathematical programs was applied to nonlinear models of the firm. The interpretation of the necessary conditions to the primal and dual problem formulations were viewed in a neoclassical economic context and a contemporary organizational context. The decision rules represented by these interpretations characterized optimal behavior. Based upon the presentation of several linear models of decentralized planning procedures, a nonlinear model of decentralized planning procedures was proposed. An interpretation of the Created Response Surface formulation of that model provided some insights into a contemporary view of decentralized organizational behavior.

APPENDIX A: MATHEMATICAL PROGRAMMING - THEOREMS AND DEFINITIONS

1. A function is said to be $\begin{pmatrix}\text{convex}\\ \text{concave}\end{pmatrix}$ over a convex set if $\bar{x}_1, \bar{x}_2 \varepsilon S$

$$f(\lambda(\bar{x}_1)+(1-\lambda)\bar{x}_2) \begin{pmatrix}\leq\\ \geq\end{pmatrix} \lambda f(\bar{x}_1) + (1-\lambda)\ f(\bar{x}_2)$$

$$0 \leq \lambda \leq 1$$

2. The sum of a finite number of convex functions is a convex function.

3. The reciprocal of a concave function is a convex function.

4. If $f(\bar{x})$ is a convex function, then $-f(\bar{x})$ is a concave function.

5. Let $g_i(x)$, $i = 1,\ldots,I$ be a $\begin{pmatrix}\text{convex}\\ \text{concave}\end{pmatrix}$ functions. Then the set S defined by:

$$S \equiv \{\ \bar{x}\ |\ g_i(x) \begin{pmatrix}\leq\\ \geq\end{pmatrix} b_i\ \}$$

is a convex set.

6. The convex problem is defined to be the $\begin{pmatrix}\text{minimization}\\ \text{maximization}\end{pmatrix}$ of a $\begin{pmatrix}\text{convex}\\ \text{concave}\end{pmatrix}$ function over a convex set.

7. Every local $\begin{pmatrix}\text{minimum}\\ \text{maximum}\end{pmatrix}$ of a $\begin{pmatrix}\text{convex}\\ \text{concave}\end{pmatrix}$ function over a convex set is also a global $\begin{pmatrix}\text{minimum}\\ \text{maximum}\end{pmatrix}$.

8. A function $L(\bar{x},\bar{\lambda}_0) \leq L(\bar{x}_0, \lambda_0) \leq L(\bar{x}_0,\bar{\lambda})$.

9. Kuhn-Tucker constraint qualification:

   Let $\bar{x}$ satisfy $\{g_i(x) \geq 0,\ i = 1,\ldots,I\}$ and let dx be any vector differential such that $\nabla g_i(\bar{x})dx \geq 0\ \forall i$ such

$g_i(\bar{x}) = 0$; then dx is tangent to some arc contained in the set of all $\bar{x}$ satisfying $\{g_i(\bar{x}) \geq 0\}$.

10. The Kuhn-Tucker necessary conditions for the existence of a saddle point to the convex problem are also sufficient conditions.

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DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

A NONLINEAR MATHEMATICAL PROGRAMMING APPROACH TO ACTIVITY ANALYSIS AND DECENTRALIZED PLANNING PROCEDURES

4. DESCRIPTIVE NOTES (Type of report and, inclusive dates)

Master's Thesis; September 1971

5. AUTHOR(S) (First name, middle initial, last name)

Ray D. Spinosa

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
September 1971	84	32

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Naval Postgraduate School Monterey, California 93940

13. ABSTRACT

A review of some of the current literature pertaining to this thesis is conducted. The Created Response Surface Technique of solving a class of nonlinear mathematical programs is presented. The theoretical interpretations of the primal and dual formulations of the technique in an activity analysis context are developed. The applicability of these interpretations to the neoclassical theory of the firm and the contemporary organization theory is indicated. Computational experience in solving well defined numerical problems is also indicated. Several linear models of organizational decentralized planning are reviewed. A nonlinear model of decentralized planning procedures is proposed and solved using the Created Response Surface Technique.

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Economics						
Mathematical Programming						

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Thesis
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A nonlinear mathematical programming approach to activity analysis and decentralized planning procedures.